Journal of Applied Mechanics and Technical Physics, Vol. 41, No. 6, 2000

CALCULATION OF LAYERED SHELLS BY THE PSEUDOGEOMETRICAL-NONLINEARITY METHOD

R. A. Kayumov

UDC 539.3

The problem of determining the stress–strain state of a multilayered shell is solved. It is assumed that the layer material is nonlinearly elastic and the strain–displacement relations are nonlinear. The displacements are expanded in terms of the functions of transverse coordinate that contain unknown parameters. The governing equations are derived with the use of the Lagrange variational principle. A technique for minimizing the energy functional is proposed. An example of a three-layered beam is considered, calculation results are compared with the exact solution, and the specific features of the approach proposed are analyzed.

Refined theories of plates and shells were considered in many studies. In constructing the theories, the series expansions in various systems of functions, asymptotic integration, and methods based on various hypotheses are used (these approaches have been reviewed in many studies, e.g., in [1–5]). Multilayered thin-walled structural elements whose mechanical properties are inhomogeneous over the thickness have broad applications in engineering; it is, therefore, necessary to refine the hypotheses on the stress and strain distribution in shells. These hypotheses are introduced for the entire package or for each layer, which leads to the fact that the order of the system of equations depends on the number of layers. One shortcoming of this approach is that it is difficult to estimate the error of solution and, hence, to compare the different shell models. However, generally, these problems can be solved by calculating and comparing the total potential energy in different theories.

In contrast to the existing approaches, we seek the components of the displacement vector in the form of a sum of the products of desired functions. As a result, pseudogeometrical nonlinearities arise in the problem. The approach proposed in this paper allows one to choose the best (in the above-mentioned sense) hypothesis on the displacement distribution in multilayered shells.

1. We consider a multilayered shell consisting of nonlinearly elastic layers. Let there be a reference surface with coordinate axes x^1 and x^2, and let the x^3 axis be normal to this surface.

We seek the components of the displacement vector u_β in the layer or in the entire package in the form of a series expansion into the functions $f^i_\beta(x^3)$:

$$u_\beta = u_{\beta 0}(x^1, x^2) + \sum_{i=1}^{I} f^i_\beta(x^3) u_{\beta i}(x^1, x^2) \quad [f^i_\beta(0) = 0, \quad \beta = 1,\, 2,\, 3]. \tag{1}$$

In most cases, especially for homogeneous shells, the first two terms are retained in the series (1), and the functions $f^i_\beta(x^3)$ are the power functions of variable x^3. In some studies, the Legendre polynomials are used. When layered shells are analyzed, the piecewise functions are often used as $f^i_\beta(x^3)$ whose parameters are determined *a priori* under conditions of rigid contact between the layers and the continuous transverse

Kazan' State Architectural Academy, Kazan' 420043. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 41, No. 6, pp. 144–152, November–December, 2000. Original article submitted May 17, 1999; revision submitted September 30, 1999.

shear and normal stresses. However, since the problem is usually solved approximately for $u_{\beta j}(x^1, x^2)$, the equations of equilibrium are not, in general, satisfied exactly at the layer boundaries. In this paper, the functions $f^i_\beta(x^3)$ are specified with accuracy to certain constants $C_{m\beta}$, which are determined in the solution of the problem. It is assumed that these functions satisfy the continuity conditions at the layer boundaries only relative to displacements.

We consider the linear approximation of $f^i_\beta(x^3)$ in the form

$$f^i_\beta(x^3) = \sum_{m=1}^{M} C_{m\beta} f^{im}_\beta(x^3) \qquad (i = 1, \ldots, I). \tag{2}$$

To determine the parameters $C_{m\beta}$ and the functions $u_{\beta i}(x^1, x^2)$ uniquely, one should impose normalizing conditions. For example, one can assume that the magnitude of the vector $\boldsymbol{C} = \{C_{1\beta}, \ldots, C_{M\beta}\}$ is equal to unity:

$$C^2_{1\beta} + C^2_{2\beta} + \ldots + C^2_{M\beta} = 1. \tag{3}$$

In deriving the governing equations for unknowns, this condition is taken into account by the Lagrange multiplier method.

To simplify calculations, one can use the condition

$$C_{1\beta} = 1. \tag{4}$$

The displacements are written in the form

$$u_\beta = u_{\beta 0}(x^1, x^2) + u_{\beta 1}(x^1, x^2)[f^{11}_\beta(x^3) + C_{2\beta} f^{12}_\beta(x^3) + \ldots + C_{M\beta} f^{1M}_\beta(x^3)]$$

$$+ u_{\beta 2}(x^1, x^2)[f^{21}_\beta(x^3) + C_{2\beta} f^{22}_\beta(x^3) + \ldots + C_{M\beta} f^{2M}_\beta(x^3)] + \ldots\ .$$

Here the functions $u_{\beta i}$ and the parameters $C_{m\beta}$ are to be determined and the functions f^{ij}_β are known.

If the displacements are not small, the strains are given by

$$2\varepsilon_{\alpha_\beta} = \nabla_\alpha u_\beta + \nabla_\beta u_\alpha + \nabla_\alpha u^\gamma \nabla_\gamma u_\beta,$$

where ∇_α denotes covariant differentiation, and summation is performed over repeated superscripts and subscripts (Greek letters).

The physical relations for a nonlinearly elastic material can be written in the form

$$\sigma^{\alpha\beta} = \frac{\partial F(I_1, \ldots, I_L)}{\partial \varepsilon_{\alpha\beta}}.$$

Here F is the elastic potential, which depends on the invariants $I_1, \ldots, I_L$ of the type of convolutions of the components of the strain tensors, the metric tensor, and the tensors of the mechanical characteristics of material.

2. The equations for the desired functions $u_{\beta i}$ and parameters $C_{m\beta}$ can be obtained with the use of different variational principles. For simplicity, we use the Lagrange principle

$$\iiint\limits_V \sigma^{\alpha\beta} \delta\varepsilon_{\alpha\beta}\, dV = \iiint\limits_V q^\alpha \delta u_\alpha\, dV + \iint\limits_S p^\alpha \delta u_\alpha\, dS. \tag{5}$$

Here V is the volume occupied by the shell, q^α are the components of the body force, and S is the surface at which the external surface loads p^α are specified.

One can see from (1) and (2) that the problem is nonlinear even in the absence of physical and geometrical nonlinearities. Therefore, we call the above approach the pseudogeometrical-nonlinearity method. To linearize the problem, we use the following procedure. In the $(n-1)$th step, the functions $u^{(n-1)}_{\beta i}$ and the parameters $C^{(n-1)}_{m\beta}$ are assumed to be known (hereinafter, the bracketed superscript denotes iteration). In the next step, the unknowns are expressed in the form

$$u_{\beta i}^{(n)} = u_{\beta i}^{(n-1)} + \Delta u_{\beta i}, \quad C_{m\beta}^{(n)} = C_{m\beta}^{(n-1)} + \Delta C_{m\beta} \quad (i = 1, \dots, I, \quad m = 2, \dots, M), \tag{6}$$

where Δ is the increment.

Setting $\Delta u_{\beta i}$, we obtain equations for $\Delta C_{m\beta} = 0$. We write the displacement, strain, and stress relations in the form

$$\varepsilon_{\alpha\beta}^{(n)} = \varepsilon_{\alpha\beta}^{(n-1)} + \Delta\varepsilon_{\alpha\beta}, \qquad \sigma_{(n)}^{\alpha\beta} = \sigma_{(n-1)}^{\alpha\beta} + E_{(n-1)}^{\alpha\beta\gamma\theta}\Delta\varepsilon_{\gamma\theta}. \tag{7}$$

Here

$$2\varepsilon_{\alpha\beta}^{(n-1)} = \nabla_\alpha u_\beta^{(n-1)} + \nabla_\beta u_\alpha^{(n-1)} + \nabla_\alpha u_{(n-1)}^\gamma \nabla_\gamma u_\beta^{(n-1)},$$

$$2\Delta\varepsilon_{\alpha\beta} = \nabla_\alpha \Delta u_\beta + \nabla_\beta \Delta u_\alpha + \nabla_\alpha u_{(n-1)}^\gamma \nabla_\gamma \Delta u_\beta + \nabla_\alpha \Delta u^\gamma \nabla_\gamma u_\beta^{(n-1)},$$

$$u_\beta^{(n-1)} = u_{\beta 0}^{(n-1)} + \sum_{i=1}^{I} u_{\beta i}^{(n-1)} \Big[\sum_{m=1}^{M} C_{m\beta}^{(n-1)} f_\beta^{im}\Big], \quad \Delta u_\beta = \Delta u_{\beta 0} + \sum_{i=1}^{I} \Delta u_{\beta i} \Big[\sum_{m=1}^{M} C_{m\beta}^{(n-1)} f_\beta^{im}\Big],$$

$$\sigma_{(n-1)}^{\alpha\beta} = \frac{\partial F^{(n-1)}}{\partial \varepsilon_{\alpha\beta}}, \qquad E_{(n-1)}^{\alpha\beta\gamma\theta} = \frac{\partial^2 F^{(n-1)}}{\partial \varepsilon_{\alpha\beta}\, \partial \varepsilon_{\gamma\theta}}.$$

For Δu_β, the variational equation (5) becomes

$$\iiint\limits_V E_{(n-1)}^{\alpha\beta\gamma\theta} \Delta\varepsilon_{\gamma\theta} \delta\Delta\varepsilon_{\alpha\beta}\, dV = -\iiint\limits_V \sigma_{(n-1)}^{\alpha\beta} \delta\Delta\varepsilon_{\alpha\beta}\, dV + \iiint\limits_V q^\beta \delta\Delta u_\beta\, dV + \iint\limits_S p^\beta \delta\Delta u_\beta\, dS. \tag{8}$$

After $\Delta u_{\beta i}$ are found from this equation, formulas (7) can be used to calculate the functions $\sigma_{(n)}^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}^{(n)}$. A system of equations for the desired parameters $\Delta C_{m\beta}$ is obtained from (5) and (6) if one assumes that $\Delta u_{\beta i} = 0$ or the functions $u_{\beta i}^{(n)} = u_{\beta i}^{(n-1)} + \Delta u_{\beta i}$ are known. In the last case, we obtain

$$u_\beta = u_{\beta 0}^{(n)}(x_1, x_2) + \sum_{i=1}^{I} u_{\beta i}^{(n)}(x_1, x_2) \sum_{m=1}^{M} (C_{m\beta}^{(n-1)} + \Delta C_{m\beta}) f_\beta^{im}(x^3). \tag{9}$$

Using the normalizing condition (4), we write the strains and stresses in the form

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}^{(n)} + \sum_{m=2}^{M} e_{\alpha\beta}^m \Delta C_{m\beta},$$

$$\begin{aligned} 2e_{\alpha\beta}^m = {} & \nabla_\alpha \Big[\sum_{i=1}^{I} f_\beta^{im} u_{\beta i}^{(n)}\Big] + \nabla_\beta \Big[\sum_{i=1}^{I} f_\alpha^{im} u_{\alpha i}^{(n)}\Big] \\ & + \nabla_\alpha \Big[u_{\gamma 0}^{(n)} + \sum_{i=1}^{I}\sum_{j=2}^{M} C_{j\beta}^{(n-1)} f_\gamma^{im} u_{\gamma i}^{(n)}\Big] \nabla^\gamma \Big[\sum_{i=1}^{I} f_\beta^{im} u_{\beta i}^{(n)}\Big] \\ & + \nabla_\alpha \Big[\sum_{i=1}^{I} f_\gamma^{im} u_{\gamma i}^{(n)}\Big] \nabla^\gamma \Big[u_{\beta 0}^{(n)} + \sum_{i=1}^{I}\sum_{j=2}^{M} C_{j\beta}^{(n-1)} f_\beta^{im} u_{\beta i}^{(n)}\Big], \end{aligned} \tag{10}$$

$$\sigma^{\alpha\beta} = \sigma_{(n)}^{\alpha\beta} + E_{(n)}^{\alpha\beta\gamma\theta} \sum_{m=2}^{M} e_{\gamma\theta}^m \Delta C_{m\beta}.$$

Substitution of (10) into (5) yields the equation for $\Delta C_{m\beta}$:

$$\iiint\limits_V \left[\sigma^{\alpha\beta}_{(n)} + E^{\alpha\beta\gamma\theta}_{(n)} \sum_{m=2}^{M} e^m_{\gamma\theta} \Delta C_{m\beta}\right] \sum_{m=2}^{M} e^m_{\gamma\theta} \delta\Delta C_{m\beta}\, dV$$

$$= \iiint\limits_V q^\beta \sum_{i=1}^{I} u^{(n)}_{\beta i} \sum_{m=2}^{M} f^{im}_\beta \delta\Delta C_{m\beta}\, dV + \iint\limits_S p^\beta \sum_{i=1}^{I} u^{(n)}_{\beta i} \sum_{m=2}^{M} f^{im}_\beta \delta\Delta C_{m\beta}\, dS. \qquad (11)$$

The iterative process of solving Eqs. (8) and (11) is repeated until the increments $\Delta C_{m\beta}$ and $\Delta u_{\beta i}$ become small relative to a certain norm.

Generally, the solution by the above method does not converge if, in particular, the initial approximation of $C_{m\beta}$ is not chosen properly. From this standpoint, in solving physically nonlinear but geometrically linear problems, the minimization methods of the potential energy of an elastic shell are preferable to reduction to the nonlinear equations (8)–(11). In the geometrically nonlinear problems, the initial approximation of the functions $u_{\beta i}$ and the parameters $C_{m\beta}$ can be obtained by the step-by-step continuation method (for example, the load can be used as a continuation parameter). In the first step, the linear problem, for which the theorem of the minimum potential energy of an elastic system is valid is considered, and methods of the optimization theory are employed to find $C_{m\beta}$. As an initial approximation, one can also use the $C_{m\beta}$ values determined from the expressions for f^i_β constructed by other methods [4, 5] under the conditions that the layers are in rigid contact with one another and the stresses are continuous in the transverse direction. In the subsequent steps of solution of system (8)–(11), the increment of the continuation parameter (load) should be not too large to ensure convergence of the process. If the process diverges, it is necessary to return to the previous step and decrease the increment.

In the case of a geometrically nonlinear problem, instead of solving system (8)–(11), one can use optimization methods as well. To this end, it is necessary to formulate the minimization problem of the functional in terms of the increments $\Delta C_{m\beta}$ for fixed $u^{(n)}_{\beta i}$ and in terms of $\Delta u_{\beta i}$ for fixed $C^{(n)}_{m\beta}$. However, the use of system (8)–(11) is preferred, since, in each iteration, one should solve the additional low-order system (11) for $\Delta C_{m\beta}$.

An approach that combines two techniques can be more convenient. The solution of Eqs. (8) yields $\Delta u_{\beta i}$, and the parameters $C_{m\beta}$ are determined by minimizing the total potential energy of an elastic system Π (this approach is employed to solve the test problem). We write Π in the form

$$\Pi = \frac{1}{2} \iiint\limits_V \sigma^{\alpha\beta} \varepsilon_{\alpha\beta}\, dV - \iiint\limits_V q^\alpha u_\alpha\, dV - \iint\limits_S p^\alpha u_\alpha\, dS.$$

The minimum-value problem can be solved by the available algorithms of optimization theory, in particular, by the following algorithm. A numerical experiment is performed for a number of $C_{m\beta}$ values, i.e., the shell problem is solved and the "experimental" values of the total potential energy Π are calculated. A regression analysis is used to construct the regression function $\psi(C_{1\beta}, C_{2\beta}, \ldots, C_{M\beta})$ from these values of Π which allows one to find the $C^*_{m\beta}$ values minimizing the function ψ. In the new neighborhood of $C^*_{m\beta}$, the new set of $C_{m\beta}$ values is chosen, the numerical experiment is performed, the new regression function $\psi(C_{1\beta}, C_{2\beta}, \ldots, C_{M\beta})$ is constructed, and the minimizing values of the parameters $C^*_{m\beta}$ are determined, and so on. When the neighborhood of the $C^*_{m\beta}$ values is diminished sufficiently, we obtain, with required accuracy, the desired parameters $C_{m\beta}$ minimizing the potential energy Π.

3. For demonstration, we apply the proposed method to the following, physically and geometrically nonlinear beam problem. A beam of unit width, height $2H$, and length l that is subject to Navier-type boundary conditions (an analog of the simply supported case) is bent under sinusoidal loading. The problem admits an exact solution [6, 7] and was solved by two techniques described above. In using (3), instead of two unknowns $C_{1\beta}$ and $C_{2\beta}$, we introduced one desired parameter that satisfies condition (3): $C_{1\beta} = \sin\varphi_\beta$ and $C_{2\beta} = \cos\varphi_\beta$. When condition (4) was considered, the only desired parameter was $C_{2\beta}$ because $C_{1\beta} = 1$.

We consider the simplest approximation (1): $u_1 = f^1_1(x^3)u_{11}(x^1) = [C_{11}f^{11}_1(x^3) + C_{21}f^{12}_1(x^3)]u_{11}(x^1)$, and $u_3 = f_{30}(x^1)$, where $f^{11}_1 = x^3$. We introduce the following notation: $x = x^1$, $z = x^3$, $f(z) = f^{12}_1(x^3)$,

$\varepsilon = \varepsilon_{11}$, $\gamma = \varepsilon_{13}$, $\sigma = \sigma^{11}$, $\tau = \sigma^{13}$, $u_{11}(x^1) = Uu(x)$, $f_{30}(x^1) = Ww(x)$, $C_1 = C_{11}$, $C_2 = C_{21}$, $\varphi = \varphi_1$, $G = G_{13}$ is the shear modulus, and $E = E_{11}$ is Young's modulus (the prime denotes differentiation). Then,

$$u_1 = [C_1 z + C_2 f(z)]Uu(x), \qquad u_3 = Ww(x), \tag{12}$$

where U, W, C_1, and C_2 are the desired constants. The strains and stresses are given by $\varepsilon = Uu'_x[C_1 z + C_2 f(z)]$, $\gamma = Uu[C_1 + C_2 f'_z(z)] + Ww'_x$, $\sigma = E\varepsilon$, and $\tau = G\gamma$.

It is assumed that the package has a symmetric structure and the load has only the normal component $q = (q_0/2)\sin(\pi x/l)$ applied to the upper and lower surfaces of the beam. In (12), one can assume that $u = \cos(\pi x/l)$ and $w = \sin(\pi x/l)$.

To simplify the problem, in the physically linear case, in Eq. (8) we pass from increments to total displacements, because $\sigma^{\alpha\beta}_{(n)} = E^{\alpha\beta\gamma\theta}\varepsilon^{(n)}_{\gamma\theta}$, $\delta\Delta\varepsilon^{(n)}_{\alpha\beta} = \delta(\varepsilon^{(n-1)}_{\alpha\beta} + \Delta\varepsilon_{\alpha\beta}) = \delta\varepsilon^{(n)}_{\alpha\beta}$, and $\delta\Delta u_\alpha = \delta u^{(n)}_\alpha$.

From (8) and (11), we obtain a system of equations for $U^{(n)}$, $W^{(n)}$, and $\Delta\varphi$ or ΔC_2. Dropping the superscripts (n) at $U^{(n)}$ and $W^{(n)}$ and the superscripts $(n-1)$ at $\varphi^{(n-1)}$ or $C_2^{(n-1)}$, we write the system in the form

$$\int_0^l \int_{-H}^H \{U(C_1 z + C_2 f)^2 (u'_x)^2 E + G[Ww'_x + Uu(C_1 + C_2 f'_z)](C_1 + C_2 f'_z)u\}\,dx\,dz = 0,$$

$$\int_0^l \int_{-H}^H \{G[Ww'_x + Uu(C_1 + C_2 f'_z)]w'_x\}\,dx\,dz = \int_0^l 2qw\,dx, \tag{13}$$

$$\int_0^l \int_{-H}^H \{U^2 E[C_1 z + C_2 f + \Delta C_1 z + \Delta C_2 f](u'_x)^2 f + WUGuw'_x f'_z$$

$$+ U^2 G[C_1 + C_2 f'_z + \Delta C_1 + \Delta C_2 f'_z] f'_z u^2\}dx\,dz = 0.$$

Here $C_1 = \sin\varphi$, $\Delta C_1 = \cos\varphi\Delta\varphi$, $C_2 = \cos\varphi$, and $\Delta C_2 = -\sin\varphi\Delta\varphi$ for the normalizing condition (3) and $\Delta C_1 = 0$ for condition (4).

We introduce the following notation:

$$J_1 = \int_{-H}^H G(z)[C_1 z + C_2 f'_z(z)]\,dz,$$

$$J_2 = \int_{-H}^H E(z)[C_1 z + C_2 f(z)]^2\,dz, \quad J_3 = \int_{-H}^H G(z)[C_1 z + C_2 f'_z(z)]^2\,dz,$$

$$J_4 = \int_{-H}^H E(z)[f'_z(z)]^2\,dz, \quad J_5 = \int_{-H}^H G(z)[f'_z(z)]^2\,dz, \quad J_6 = \int_{-H}^H G(z)\,dz, \tag{14}$$

$$J_7 = \int_{-H}^H E(z) z f(z)\,dz, \quad J_8 = \int_{-H}^H G(z) f'_z(z)\,dz, \quad J_9 = \int_{-H}^H E(z) f^2(z)\,dz.$$

TABLE 1

Solution	$u_1E_1/(2q_0H)$	$u_3E_1/(2q_0H)$	σ^{11}/q_0
Exact solution	11.015	57.10	17.40
(15), (16)	11.044	57.32	17.35
	(0.26)	(0.38)	(0.29)
[6]	10.455	54.8	17.55
	(3.9)	(4.4)	(0.68)

Note. The error Δ (in percent) is given in brackets.

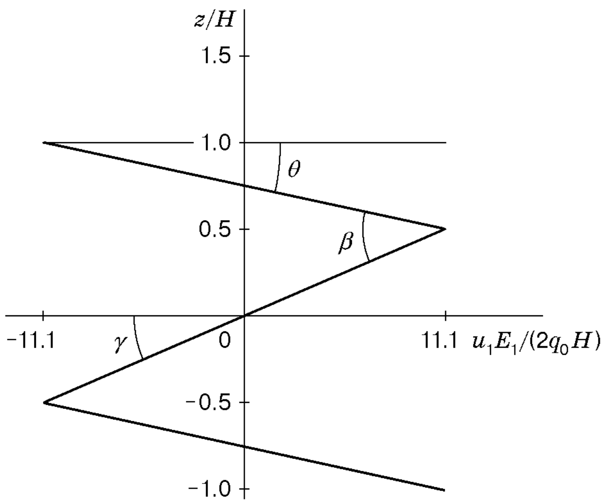

Fig. 1

With allowance for (14), from (13) we obtain the solution

$$U = q_0 l J_2/a, \quad a = \pi(J_2^2 - \pi^2 J_1 J_2/l^2 - J_3 J_6), \quad W = -q_0(\pi^2 J_1 + l^2 J_3)/(\pi a),$$
$$\Delta\varphi = \frac{J_8(\pi^2 J_1 + l^2 J_3)/J_2 - (C_1 J_7 + C_2 J_9)\pi^2 - (C_1 J_8 + C_2 J_5) l^2}{(C_2 J_7 - C_1 J_9)\pi^2 + (C_2 J_8 - C_1 J_5) l^2}; \qquad (15)$$

$$\Delta C_2 = [(\pi^2 J_1 + l^2 J_3) J_5 - \pi^2 J_2 (J_7 + C_2 J_4) - l^2 J_2 (J_8 + C_2 J_5)]/[(\pi^2 J_4 + l^2 J_5) J_2]. \qquad (16)$$

It follows from (15) that in this problem, the parameter C_2 or φ can be determined by the iterative method independently of U and W.

Below, we give numerical results for the case of a three-layered beam [6]. It is assumed that $h_1 = h_3 = h_2/2 = h = H/2$ (the subscripts 1, 2, and 3 are the layer numbers, beginning from the lower layer), $4H = l$, $E_1/E_2 = E_3/E_2 = 500$, $G_1/G_2 = G_3/G_2 = 500$, and $G_i = E_i/2.6$. The function $f(z)$ is taken to be

$$f = \begin{cases} z - h, & h \leqslant z \leqslant H, \\ 0, & -h \leqslant z \leqslant h, \\ z + h, & -H \leqslant z \leqslant -h. \end{cases} \qquad (17)$$

Owing to the symmetry, we can decrease the number of unknowns $C_{m\beta}$ by choosing the function $f(z)$ in such a manner that it determines the law of change of the displacements in the group of two external layers. As a result, with allowance for condition (4), the integrals (14) become

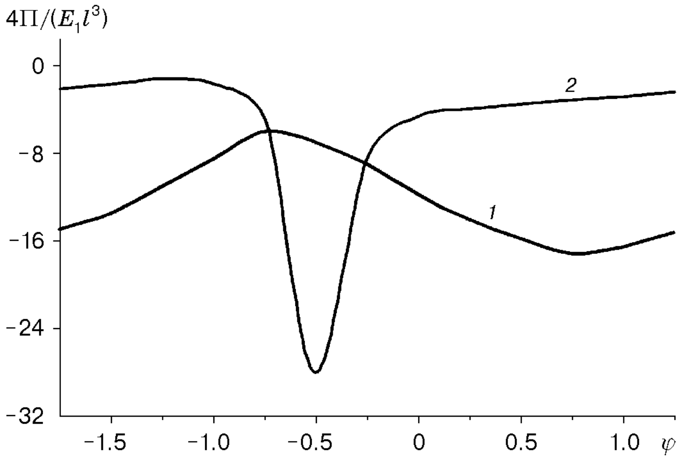

Fig. 2

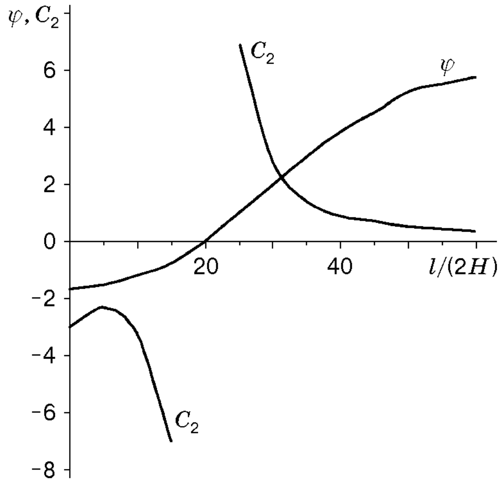

Fig. 3

$$J_1 = 2h^3[E^2 + 7E_1 + 5C_2E_1 + C_2^2E_1]/3, \quad J_2 = 2h[G_2 + G_1(1 + C_2)], \quad J_3 = 2h[G_2 + G_1C_2^2],$$

$$J_4 = 2h^3E_1/3, \quad J_5 = 2hG_1, \quad J_6 = 2h[G_2 + G_1], \quad J_7 = 5J_4/2, \quad J_8 = J_5.$$

In this problem, C_2 can be found by solving Eq. (16) by iterations or reducing it to a quadratic equation for C_2 by setting $\Delta C_2 = 0$ in (16). Calculations give $C_2 = -3.032$. Table 1 lists the dimensionless values of the displacements u_1 at the angular point, the displacements u_3 at the center of the beam, and the maximum stress σ^{11}. The exact values that can be obtained from [6] are given in the first row of the table, the calculation results obtained from formulas (15) and (16) are given in the second row, and the results obtained in [6] are given in the third row. In minimizing the total potential energy of the beam with one unknown parameter φ (or C_2), one can use simple methods. In this problem, the bisection method was preferred to the method of quadratic approximation of the function Π.

The continuous distribution of transverse shear stresses over the beam thickness can be obtained by integrating the equation of equilibrium $\partial\sigma/\partial x + \partial\tau/\partial z = 0$ provided $\sigma(x, z)$ is known. Since the rough approximation in the form of a piecewise-constant function of variable z was used to calculate the transverse shear strains, the accuracy of determination of τ is not high. For example, we obtain $\tau_{\max}/q_0 = 1.73$ in the first and third layers, whereas the exact value is 2.71 [6].

Figure 1 shows the diagram of the tangential displacements $u_1E_1/(2q_0H)$ for $x = 0$. The break angle $\pi - \beta$ is expressed in terms of C_1 and C_2 as follows:

$$\pi - \beta = \pi - \theta - \gamma = \pi - \arctan\,(k(C_1 + C_2)) - \arctan\,(kC_1), \quad k = UE_1/(2q_0H).$$

Here $C_1 = 1$ or $C_1 = \sin\varphi$ and $C_2 = \cos\varphi$ for condition (4). Thus, C_2 or φ determine the degree of break of the normal.

The results of the numerical experiments have revealed the following specific features of the problem.

1. The convergence of the method depends on the initial approximation of the parameters C_2 or φ and on the mechanical characteristics of the layers. Figure 2 shows qualitative dependences of $4\Pi/(E_1l^3)$ on the parameter φ. Curve 1 refers to the isotropic beam ($E_1 = E_2$ and $G_1 = G_2$; tenfold values of Π are shown), and curve 2 refers to the above-considered case. One can see from Fig. 2 that, in the case of a layered shell made from materials with different mechanical characteristics, small changes in the value of φ (small variations in the displacement distribution), which is specified *a priori* and is not refined in the iterative process, can lead to values of Π that differ considerably from $\Pi_{\min}$. In contrast, in the case of an isotropic shell, even large errors in the displacement distribution over the thickness do not lead to significant errors in determining stresses and displacements. If system (8)–(11) is used, one can obtain $\Pi_{\max}$ instead of $\Pi_{\min}$ because of a poor choice of the initial value of φ.

2. When condition (4) is used, some of the unknowns $C_2, \ldots, C_M$ considered as functions of certain parameters (say, relative thickness) can have discontinuities of the second kind. Figure 3 shows the dependences $\varphi(l/(2H))$ and $C_2(l/(2H))$ at the above values of the elastic constants. The discontinuity of the second kind in the function C_2 is attributed to the fact that in this problem, the contribution of the function $f_1^{11} = z$ to f_1^1 can be decreased only by increasing $|C_2|$. The discontinuity has no effect on the values of u_1, u_3, σ, or ε. However, to avoid these discontinuities, one should take f_j^{ik} such that the class of functions f_j^i is as broad as possible. In the problem considered, it is sufficient to take the function $f(z)$ in the form

$$f(z) = \begin{cases} a(z-h)+h, & z \geqslant h, \\ z, & -h \geqslant z \geqslant h, \\ a(z+h)-h, & -h \geqslant z, \end{cases}$$

where $a \neq 1$.

Another method is to use condition (3), but this complicates the problem, except for the case where $M = 2$.

It is noteworthy that a similar analysis can be performed in the cases where the Reissner-type mixed functionals are used.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 99-01-00410).

REFERENCES

1. S. A. Ambartsumyan, *Theory of Anisotropic Plates* [in Russian], Nauka, Moscow (1987).
2. Kh. Al'tenbakh, "The main trends in the theory of multilayered thin-walled structures (review)," *Mekh. Kompoz. Mater.*, **34**, No. 6, 333–348 (1998).
3. E. I. Grigolyuk and G. M. Kulikov, "Development of the general trend in the theory of multilayered shells," *Mekh. Kompoz. Mater.*, No. 2, 287–298 (1998).
4. A. A. Dudchenko, S. A. Lur'e, and I. F. Obraztsov, "Anisotropic multilayered plates and shells," *Itogi Nauki Tekh., Ser. Mekh. Deform. Tverd. Tela*, **15**, 3–68 (1983).
5. Ya. M. Grigorenko and A. T. Vasilenko, "The methods of solving the static problems of shells with a uniform structure," *Prikl. Mekh.*, **34**, No. 10, 42–49 (1998).
6. V. K. Prisyazhnyuk and I. B. Zaivelev, "Solution of the plane elastic problem for a multilayered orthotropic composite," *Mekh. Kompoz. Mater.*, No. 2, 206–214 (1991).
7. R. M. Christensen, *Mechanics of Composite Materials*, John Wiley and Sons, New York (1979).